A Counter-Example to the Mismatched Decoding Converse for Binary-Input Discrete Memoryless Channels

Jonathan Scarlett, Anelia Somekh-Baruch, Alfonso Martinez, and Albert Guillén i Fàbregas

***Abstract*—This paper studies the mismatched decoding problem for binary-input discrete memoryless channels. An example is provided for which an achievable rate based on superposition coding exceeds the the LM rate (Hui, 1983; Csiszár-Körner, 1981), thus providing a counter-example to a previously reported converse result (Balakirsky, 1995). Both numerical evaluations and theoretical results are used in establishing this claim.**

I. Introduction

In this paper, we consider the problem of channel coding with a given (possibly suboptimal) decoding rule, i.e. mismatched decoding [1]–[4]. This problem is of significant interest in settings where the optimal decoder is ruled out due to channel uncertainty or implementation constraints, and also has several connections to theoretical problems such as zero-error capacity. Finding a single-letter expression for the channel capacity with mismatched decoding is a long-standing open problem, and is believed to be very difficult; the vast majority of the literature has focused on achievability results. The only reported single-letter converse result for general decoding metrics is that of Balakirsky [5], who considered binary-input discrete memoryless channels (DMCs) and stated a matching converse to the achievable rate of Hui [1] and Csiszár-Körner [2]. However, in the present paper, we provide a counter-example to this converse, i.e. a binary-input DMC for which this rate can be exceeded.

We proceed by describing the problem setup. The encoder and decoder share a codebook $\mathcal{C} = \{\boldsymbol{x}^{(1)}, \dots \boldsymbol{x}^{(M)}\}$ containing M codewords of length n. The encoder receives a message m equiprobable on the set $\{1, \dots M\}$ and transmits $\boldsymbol{x}^{(m)}$.

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The output sequence $\boldsymbol{y}$ is generated according to $W^n(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^n W(y_i|x_i)$, where W is a single-letter transition law from $\mathcal{X}$ to $\mathcal{Y}$. The alphabets are assumed to be finite, and hence the channel is a DMC. Given the output sequence $\boldsymbol{y}$, an estimate of the message is formed as follows:

$$\hat{m} = \arg\max_j q^n(\boldsymbol{x}^{(j)}, \boldsymbol{y}), \tag{1}$$

where $q^n(\boldsymbol{x}, \boldsymbol{y}) \triangleq \prod_{i=1}^n q(x_i, y_i)$ for some non-negative function q called the *decoding metric*. An error is said to have occurred if $\hat{m}$ differs from m, and the error probability is denoted by

$$p_{\mathrm{e}} \triangleq \mathbb{P}[\hat{m} \neq m]. \tag{2}$$

We assume that ties are broken as errors. A rate R is said to be achievable if, for all $\delta > 0$, there exists a sequence of codebooks with $M \geq e^{n(R-\delta)}$ codewords having vanishing error probability under the decoding rule in (1). The mismatched capacity of (W, q) is defined to be the supremum of all achievable rates, and is denoted by C_{M}.

In this paper, we focus on binary-input DMCs, and we will be primarily interested in the achievable rates based on constant-composition codes due to Hui [1] and Csiszár and Körner [2], an achievable rate based on superposition coding by the present authors [6]–[8], and a reported converse by Balakirsky [5]. These are introduced in Sections I-B and I-C.

A. Notation

The set of all probability mass functions (PMFs) on a given finite alphabet, say $\mathcal{X}$, is denoted by $\mathcal{P}(\mathcal{X})$, and similarly for conditional distributions (e.g. $\mathcal{P}(\mathcal{Y}|\mathcal{X})$). The marginals of a joint distribution $P_{XY}(x,y)$ are denoted by $P_X(x)$ and $P_Y(y)$. Similarly, $P_{Y|X}(y|x)$ denotes the conditional distribution induced by $P_{XY}(x,y)$. We write $P_X = \widetilde{P}_X$ to denote element-wise equality between two probability distributions on the same alphabet. Expectation with respect to a distribution $P_X(x)$ is denoted by $\mathbb{E}_P[\cdot]$. Given a distribution $Q(x)$ and a conditional distribution $W(y|x)$, the joint distribution $Q(x)W(y|x)$ is denoted by $Q \times W$. Information-theoretic quantities with respect to a given distribution (e.g. $P_{XY}(x,y)$) are written using a subscript (e.g. $I_P(X;Y)$). All logarithms have base e, and all rates are in nats/use.

B. Achievability

The most well-known achievable rate in the literature, and the one of the most interest in this paper, is the LM rate, which is given as follows for an arbitrary input distribution $Q \in \mathcal{P}(\mathcal{X})$:

$$I_{\mathrm{LM}}(Q) \triangleq \min_{\substack{\widetilde{P}_{XY} \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \,:\, \widetilde{P}_X = Q,\, \widetilde{P}_Y = P_Y \\ \mathbb{E}_{\widetilde{P}}[\log q(X,Y)] \geq \mathbb{E}_P[\log q(X,Y)]}} I_{\widetilde{P}}(X;Y), \tag{3}$$

where $P_{XY} \triangleq Q \times W$. This rate was derived independently by Hui [1] and Csiszár-Körner [2]. The proof uses a standard random coding construction in which each codeword is independently drawn according to the uniform distribution on a given type class. The following alternative expression was given by Merhav *et al.* [4] using Lagrange duality:

$$I_{\mathrm{LM}}(Q) \triangleq \sup_{s \geq 0, a(\cdot)} \sum_{x,y} Q(x) W(y|x) \log \frac{q(x,y)^s e^{a(x)}}{\sum_{\overline{x}} Q(\overline{x}) q(\overline{x},y)^s e^{a(\overline{x})}}. \tag{4}$$

Since the input distribution Q is arbitrary, we can optimize it to obtain the achievable rate $C_{\mathrm{LM}} \triangleq \max_Q I_{\mathrm{LM}}(Q)$. In general, C_{M} may be strictly higher than C_{LM} [2], [9].

The first approach to obtaining achievable rates exceeding C_{LM} was given in [2]. The idea is to code over pairs of symbols: If a rate R is achievable for the channel $W^{(2)}((y_1,y_2)|(x_1,x_2)) \triangleq W(y_1|x_1)W(y_2|x_2)$ with the metric $q^{(2)}((x_1,x_2),(y_1,y_2)) \triangleq q(x_1,y_1)q(x_2,y_2)$, then $\frac{R}{2}$ is achievable for the original channel W with the metric q. Thus, one can apply the LM rate to $(W^{(2)}, q^{(2)})$, optimize the input distribution on the product alphabet, and infer an achievable rate for (W,q); we denote this rate by $C_{\mathrm{LM}}^{(2)}$. An example was given in [2] for which $C_{\mathrm{LM}}^{(2)} > C_{\mathrm{LM}}$. Moreover, as stated in [2], the preceding arguments can be applied to the k-th order product channel for $k > 2$; we denote the corresponding achievable rate by $C_{\mathrm{LM}}^{(k)}$. It was conjectured in [2] that $\lim_{k\to\infty} C_{\mathrm{LM}}^{(k)} = C_{\mathrm{M}}$. It should be noted that the computation of $C_{\mathrm{LM}}^{(k)}$ is generally prohibitively complex even for relatively small values of k, since $I_{\mathrm{LM}}(Q)$ is non-concave in general [10].

Another approach to improving on C_{LM} is to use multi-user random coding ensembles exhibiting more structure than the standard ensemble containing independent codewords. This idea was first proposed by Lapidoth [9], who used parallel coding techniques to provide an example where $C_{\mathrm{M}} = C$ (with C being the matched capacity) but $C_{\mathrm{LM}} < C$. Building on these ideas, further achievable rates were provided by the present authors [6]–[8] using superposition coding techniques. Of particular interest in this paper is the following. For any finite auxiliary alphabet $\mathcal{U}$ and input distribution Q_{UX}, the rate $R = R_0 + R_1$ is achievable for any (R_0, R_1) satisfying[1]

$$R_1 \leq \min_{\substack{\widetilde{P}_{UXY} \in \mathcal{P}(\mathcal{U}\times\mathcal{X}\times\mathcal{Y}) \,:\, \widetilde{P}_{UX} = Q_{UX},\, \widetilde{P}_{UY} = P_{UY} \\ \mathbb{E}_{\widetilde{P}}[\log q(X,Y)] \geq \mathbb{E}_P[\log q(X,Y)]}} I_{\widetilde{P}}(X;Y|U) \tag{5}$$

$$R_0 \leq \min_{\substack{\widetilde{P}_{UXY} \in \mathcal{P}(\mathcal{U}\times\mathcal{X}\times\mathcal{Y}) \,:\, \widetilde{P}_{UX} = Q_{UX},\, \widetilde{P}_{Y} = P_{Y} \\ \mathbb{E}_{\widetilde{P}}[\log q(X,Y)] \geq \mathbb{E}_P[\log q(X,Y)]}} I_{\widetilde{P}}(U;X) + \big[I_{\widetilde{P}}(X;Y|U) - R_1\big]^+, \tag{6}$$

where $P_{UXY} \triangleq Q_{UX} \times W$. We define $I_{\mathrm{SC}}(Q_{UX})$ to be the maximum of $R_0 + R_1$ subject to these constraints, and we write the optimized rate as $C_{\mathrm{SC}} \triangleq \sup_{\mathcal{U}, Q_{UX}} I_{\mathrm{SC}}(Q_{UX})$. We also note the following dual expressions for (5)–(6) [6], [8]:

$$R_1 \leq \sup_{s\geq 0, a(\cdot,\cdot)} \sum_{u,x,y} Q_{UX}(u,x) W(y|x) \times \log \frac{q(x,y)^s e^{a(u,x)}}{\sum_{\overline{x}} Q_{X|U}(\overline{x}|u) q(\overline{x},y)^s e^{a(u,\overline{x})}} \tag{7}$$

$$R_0 \leq \sup_{\rho_1 \in [0,1], s \geq 0, a(\cdot,\cdot)} -\rho_1 R_1 + \sum_{u,x,y} Q_{UX}(u,x) W(y|x) \times \log \frac{\big(q(x,y)^s e^{a(u,x)}\big)^{\rho_1}}{\sum_{\overline{u}} Q_U(\overline{u}) \Big(\sum_{\overline{x}} Q_{X|U}(\overline{x}|\overline{u}) q(\overline{x},y)^s e^{a(\overline{u},\overline{x})} \Big)^{\rho_1}}. \tag{8}$$

Outlines of the derivations of both the primal and dual expressions can also be found in an extended version of this paper [11].

We note that C_{SC} is at least as high as Lapidoth's parallel coding rate [6]–[8], though it is not known whether it can be strictly higher. In [6], a refined version of superposition coding was shown to yield a rate improving on $I_{\mathrm{SC}}(Q_{UX})$ for fixed $(\mathcal{U}, Q_{UX})$, but the standard version will suffice for our purposes.

The above-mentioned technique of passing to the k-th order product alphabet is equally valid for the superposition coding achievable rate, and we denote the resulting achievable rate by $C_{\mathrm{SC}}^{(k)}$. The rate $C_{\mathrm{SC}}^{(2)}$ will be particularly important in this paper, and we will also use the analogous quantity $I_{\mathrm{SC}}^{(2)}(Q_{UX})$ with a fixed input distribution Q_{UX}. Since the input alphabet of the product channel is $\mathcal{X}^2$, one might more precisely write the input distribution as $Q_{UX^{(2)}}$, but we omit this additional superscript. The choice $\mathcal{U} = \{0,1\}$ for the auxiliary alphabet will prove to be sufficient for our purposes.

C. Converse

Very few converse results have been provided for the mismatched decoding problem. Csiszár and Narayan [3] showed that $\lim_{k\to\infty} C_{\mathrm{LM}}^{(k)} = C_{\mathrm{M}}$ for erasures-only metrics, i.e. metrics such that $q(x,y) = \max_{x,y} q(x,y)$ for all (x,y) such that $W(y|x) > 0$. More recently, multi-letter converse results were given by Somekh-Baruch [12], yielding a general formula for the mismatched capacity in the sense of Verdú-Han [13]. However, these expressions are not computable.

The only general single-letter converse result presented in the literature is that of Balakirsky [14], who reported that $C_{\mathrm{LM}} = C_{\mathrm{M}}$ for binary-input DMCs. In the following section, we provide a counter-example showing that in fact the strict inequality $C_{\mathrm{M}} > C_{\mathrm{LM}}$ can hold even in this case.

[1]The condition in (6) has a slightly different form to that in [6], which contains the additional constraint $I_{\widetilde{P}}(U;X) \leq R_0$ and replaces the $[\cdot]^+$ function in the objective by its argument. Both forms are given in [7], and their equivalence is proved therein. A simple way of seeing this equivalence is by noting that both expressions can be written as $0 \leq \min_{\widetilde{P}_{UXY}} \max\big\{I_{\widetilde{P}}(U,X;Y) - (R_0+R_1), I_{\widetilde{P}}(U;X) - R_0\big\}$.

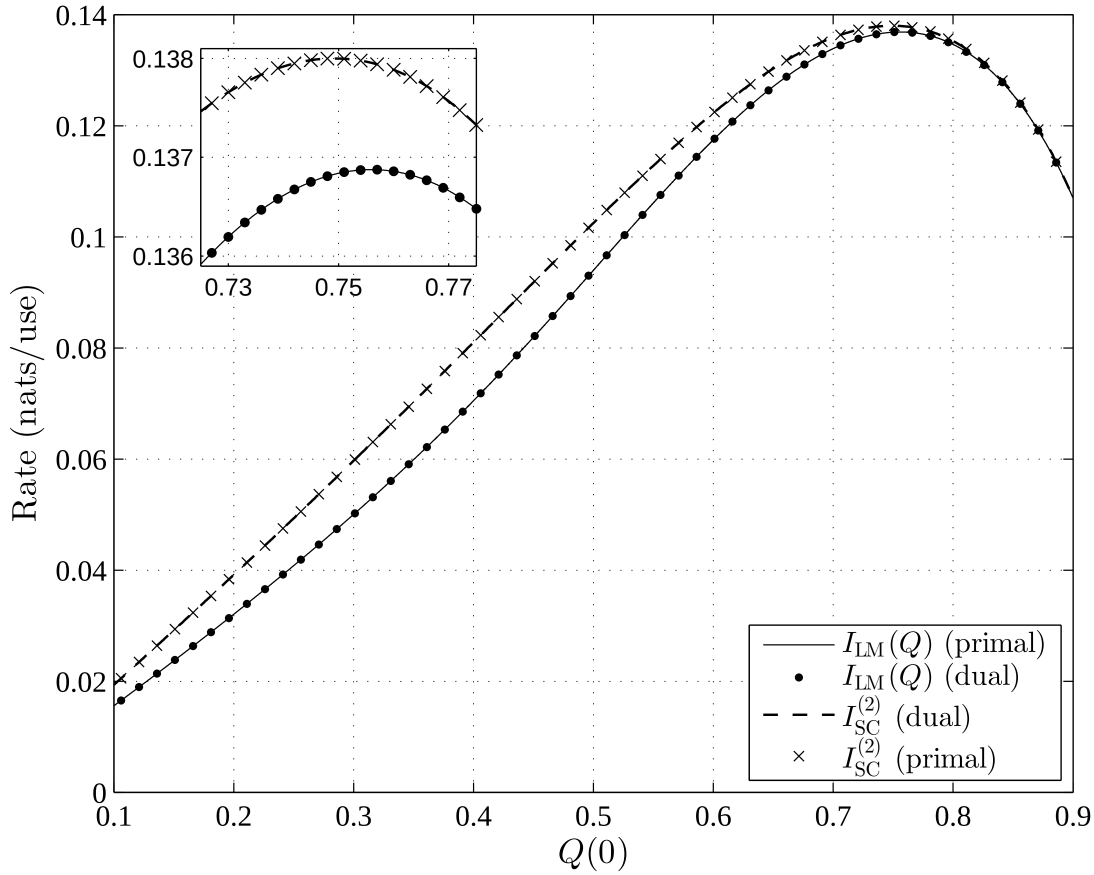

Figure 1: Numerical evaluations of the LM rate $I_{\mathrm{LM}}(Q)$ as a function of the (first entry of the) input distribution, and the corresponding superposition coding rate $I_{\mathrm{SC}}^{(2)}(Q_{UX})$ using the construction described in Section III-D. The matched capacity is $C \approx 0.4944$ nats/use, and is achieved by $Q(0) \approx 0.5398$.

II. The Counter-Example

The main claim of this paper is the following; the details are given in Section III.

Counter-Example 1. *Let $\mathcal{X} = \{0,1\}$ and $\mathcal{Y} = \{0,1,2\}$, and consider the channel and metric described by the entries of the $|\mathcal{X}| \times |\mathcal{Y}|$ matrices*

$$\boldsymbol{W} = \begin{bmatrix} 0.97 & 0.03 & 0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0.5 & 1.36 \end{bmatrix}. \quad (9)$$

Then the LM rate satisfies

$$0.136874 \le C_{\mathrm{LM}} \le 0.136900 \quad \text{nats/use}, \quad (10)$$

whereas the superposition coding rate obtained by considering the second-order product of the channel is lower bounded by

$$C_{\mathrm{SC}}^{(2)} \ge 0.137998 \quad \text{nats/use}. \quad (11)$$

Consequently, we have $C_{\mathrm{M}} > C_{\mathrm{LM}}$.

We proceed by presenting various points of discussion.

Numerical Evaluations: While (10) and (11) are obtained using numerical computations, and the difference between the two is small, we will take care in ensuring that the gap is genuine, rather than being a matter of numerical accuracy. All of the code used in our computations is available online [15].

Figure 1 plots our numerical evaluations of $I_{\mathrm{LM}}(Q)$ and $I_{\mathrm{SC}}^{(2)}(Q_{UX})$ for a range of input distributions; for the latter, Q_{UX} is determined from Q in a manner to be described in Section III-D. Note that this plot is only meant to help the reader visualize the results; it is not sufficient to establish Counter-Example 1 in itself. Nevertheless, it is reassuring to see that the curves corresponding to the primal and dual expressions are indistinguishable.

Our computations suggest that

$$C_{\mathrm{LM}} \approx 0.136875 \quad \text{nats/use}, \quad (12)$$

and that the optimal input distribution is approximately

$$Q = \begin{bmatrix} 0.75597 & 0.24403 \end{bmatrix}. \quad (13)$$

The matched capacity is significantly higher than C_{LM}, namely $C \approx 0.4944$ nats/use, with a corresponding input distribution approximately equal to $[0.5398 \; 0.4602]$. As seen in the proof, the fact that the right-hand side of (10) exceeds that of (12) by 2.5×10^{-5} is due to the use of (possibly crude) bounds on the loss in the rate when Q is slightly suboptimal.

Other Achievable Rates: One may question whether (11) can be improved by considering $C_{\mathrm{SC}}^{(k)}$ for $k > 2$. However, we were unable to find any such improvement when we tried $k = 3$; see Section III-D for further discussion on this attempt. Similarly, we observed no improvement on (12) when we computed $I_{\rm LM}^{(2)}(Q^{(2)})$ with a brute force search over $Q^{(2)} \in \mathcal{P}(\mathcal{X}^2)$ to two decimal places. Of course, it may still be that $C_{\rm LM}^{(k)} > C_{\rm LM}$ for some $k > 2$, but optimizing $Q^{(k)}$ quickly becomes computationally difficult; even for $k = 3$, the search space is 7-dimensional with no apparent convexity structure.

Our numerical findings also showed no improvement of the superposition coding rate $C_{\rm SC}$ for the original channel (as opposed to the product channel) over the LM rate $C_{\rm LM}$.

We were also able to obtain the achievable rate in (10) using Lapidoth's expurgated parallel coding rate [9] (or more precisely, its dual formulation from [6]) to the second-order product channel. In fact, this was done by taking the input distribution Q_{UX} and the dual parameters (s, a, ρ_1) used in (7)–(8) (see Section III-D), and "transforming" them into parameters for the expurgated parallel coding ensemble that achieve an identical rate. Details are given in Appendix A.

Choices of Channel and Metric: While the decoding metric in (9) may appear to be unusual, it should be noted that any decoding metric with $\max_{x,y} q(x,y) > 0$ is equivalent to another metric yielding a matrix of this form with the first row and first column equal to one [5], [14].

One may question whether the LM rate can be improved for binary-input binary-output channels, as opposed to our ternary-output example. However, this is not possible, since for any such channel the LM rate is either equal to zero or the matched capacity, and in either case it coincides with the mismatched capacity [3].

Unfortunately, despite considerable effort, we have been unable to understand the analysis given in [14] in sufficient detail to identify any major errors therein. We also remark that for the vast majority of the examples we considered, $C_{\rm LM}$ was indeed greater than or equal to all other achievable rates that we computed. However, (9) was not the only counter-example, and others were found with $\min_{x,y} W(y|x) > 0$ (in contrast with (9)). For example, a similar gap between the rates was observed when the first row of $\boldsymbol{W}$ in (9) was replaced by [0.97 0.02 0.01].

III. Establishing Counter-Example 1

While Counter-Example 1 is concerned with the specific channel and metric given in (9), we will present several results for more general channels with $\mathcal{X} = \{0,1\}$ and $\mathcal{Y} = \{0,1,2\}$ (and in some cases, arbitrary finite alphabets). To make some of the expressions more compact, we define $Q_x \triangleq Q(x)$, $W_{xy} \triangleq W(y|x)$ and $q_{xy} \triangleq q(x,y)$ throughout this section.

A. Auxiliary Lemmas

The optimization of $I_{\rm LM}(Q)$ over Q can be difficult, since $I_{\rm LM}(Q)$ is non-concave in Q in general [10]. Since we are considering the case $|\mathcal{X}| = 2$, this optimization is one-dimensional, and we thus resort to a straightforward brute-force search of Q_0 over a set of regularly-spaced points in $[0,1]$. To establish the upper bound in (10), we must bound the difference $C_{\rm LM} - I_{\rm LM}(Q_0)$ for the choice of Q_0 maximizing the LM rate among all such points. Lemma 2 below is used for precisely this purpose; before stating it, we present a preliminary result on the continuity of the binary entropy function $H_2(\alpha) \triangleq -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$.

It is well-known that for two distributions Q and Q' on a common finite alphabet, we have $|H(Q') - H(Q)| \le \delta \log \frac{|\mathcal{X}|}{\delta}$ whenever $\|Q' - Q\|_1 \le \delta$ [16, Lemma 2.7]. The following lemma gives a refinement of this statement for the case that $|\mathcal{X}| = 2$ and $\min\{Q'_0, Q'_1\}$ is no smaller than a predetermined constant.

Lemma 1. *Let $Q' \in \mathcal{P}(\mathcal{X})$ be a PMF on $\mathcal{X} = \{0,1\}$ such that $\min\{Q'_0, Q'_1\} \ge Q'_{\min}$ for some $Q'_{\min} > 0$. For any PMF $Q \in \mathcal{P}(\mathcal{X})$ such that $|Q_0 - Q'_0| \le \delta$ (or equivalently, $|Q_1 - Q'_1| \le \delta$), we have*

$$\big|H(Q') - H(Q)\big| \le \delta \log \frac{1 - Q'_{\min}}{Q'_{\min}}. \tag{14}$$

Proof. Set $\Delta \triangleq Q_0 - Q'_0$. Since $H_2(\cdot)$ is concave, the straight line tangent to a given point always lies above the function itself. Assuming without loss of generality that $Q'_0 \le 0.5$, we have

$$\big|H_2(Q'_0 + \Delta) - H_2(Q'_0)\big| \le |\Delta| \cdot \frac{dH_2}{d\alpha}\bigg|_{\alpha = Q'_0} \tag{15}$$

$$= |\Delta| \log \frac{1 - Q'_0}{Q'_0}. \tag{16}$$

The desired result follows since $\frac{1-Q'_0}{Q'_0}$ is decreasing in Q'_0, and since $Q'_0 \ge Q'_{\min}$ and $|\Delta| \le \delta$ by assumption. □

The following lemma builds on the preceding lemma, and is key to establishing Counter-Example 1.

Lemma 2. *For any binary-input mismatched DMC, we have the following under the setup of Lemma 1:*

$$I_{\rm LM}(Q) \ge I_{\rm LM}(Q') - \delta \log \frac{1 - Q'_{\min}}{Q'_{\min}} - \frac{\delta \log 2}{Q'_{\min}}. \tag{17}$$

Proof. The bound in (17) is trivial when $I_{\rm LM}(Q') = 0$, so we consider the case $I_{\rm LM}(Q') > 0$. Observing that $Q(x) > 0$ for $x \in \{0,1\}$, we can make the change of variable $a(x) = \log \frac{e^{\tilde{a}(x)}}{Q(x)}$ (i.e. $e^{\tilde{a}(x)} = Q(x)e^{a(x)}$) in (4) to obtain

$$I_{\rm LM}(Q) = \sup_{s \ge 0, \tilde{a}(\cdot)} \sum_{x,y} Q(x) W(y|x) \times \log \frac{q(x,y)^s e^{\tilde{a}(x)}}{Q(x) \sum_{\overline{x}} q(\overline{x},y)^s e^{\tilde{a}(\overline{x})}}, \tag{18}$$

which can equivalently be written as

$$I_{\rm LM}(Q') = H(Q') - \inf_{s \ge 0, \tilde{a}(\cdot)} \sum_{x,y} Q'(x) W(y|x) \times \log\left(1 + \frac{q(\overline{x},y)^s e^{\tilde{a}(\overline{x})}}{q(x,y)^s e^{\tilde{a}(x)}}\right), \tag{19}$$

where $\overline{x} \in \{0,1\}$ denotes the unique symbol differing from $x \in \{0,1\}$.

The following arguments can be simplified when the infimum is achieved, but for completeness we consider the general case. Let $(s_k, \tilde{a}_k)$ be a sequence of parameters such that

$$H(Q') - \lim_{k\to\infty} \sum_{x,y} Q'(x) W(y|x) \times \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) = I_{\mathrm{LM}}(Q'). \quad (20)$$

Since the argument to the logarithm in (20) is no smaller than one, and since $H(Q') \le \log 2$ by the assumption that the input alphabet is binary, we have for $x = 0, 1$ and sufficiently large k that

$$\sum_y Q'(x) W(y|x) \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) \le \log 2, \quad (21)$$

since otherwise the left-hand side of (20) would be non-positive, in contradiction with the fact that we are considering the case $I_{\mathrm{LM}}(Q') > 0$. Using the assumption $\min\{Q'_0, Q'_1\} \ge Q'_{\min}$, we can weaken (21) to

$$\sum_y W(y|x) \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) \le \frac{\log 2}{Q'_{\min}}. \quad (22)$$

We now have the following:

$$\begin{aligned}
&I_{\mathrm{LM}}(Q) \\
&\ge H(Q) - \limsup_{k\to\infty} \sum_{x,y} Q(x) W(y|x) \times \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) && (23)\\
&\ge H(Q') - \limsup_{k\to\infty} \sum_{x} Q(x) \sum_y W(y|x) \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) - \delta \log\frac{1-Q'_{\min}}{Q'_{\min}} && (24)\\
&= H(Q') - \limsup_{k\to\infty} \sum_{x} (Q(x) + Q'(x) - Q'(x)) \times \sum_y W(y|x) \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) - \delta \log\frac{1-Q'_{\min}}{Q'_{\min}} && (25)\\
&\ge H(Q') - \limsup_{k\to\infty} \sum_{x} Q'(x) \sum_y W(y|x) \times \log\left(1 + \frac{q(\overline{x},y)^{s_k} e^{\tilde{a}_k(\overline{x})}}{q(x,y)^{s_k} e^{\tilde{a}_k(x)}}\right) - \delta \log\frac{1-Q'_{\min}}{Q'_{\min}} - \frac{\delta \log 2}{Q'_{\min}} && (26)\\
&= I_{\mathrm{LM}}(Q') - \delta \log\frac{1-Q'_{\min}}{Q'_{\min}} - \frac{\delta \log 2}{Q'_{\min}}, && (27)
\end{aligned}$$

where (23) follows by replacing the infimum in (19) by the particular sequence of parameters $(s_k, \tilde{a}_k)$ and taking the $\limsup$, (24) follows from Lemma 1, (26) follows by applying (22) for the x value where $Q(x) \ge Q'(x)$ and lower bounding the logarithm by zero for the other x value, and (27) follows from (20). □

B. *Establishing the Upper Bound in* (10)

As mentioned in the previous subsection, we optimize Q by performing a brute force search over a set of regularly spaced points, and then using Lemma 2 to bound the difference $C_{\mathrm{LM}} - I_{\mathrm{LM}}(Q)$. We let the input distribution therein be $Q' = \arg\max_Q I_{\mathrm{LM}}(Q)$. Note that this maximum is always achieved, since I_{LM} is continuous and bounded [3]. If there are multiple maximizers, we choose one arbitrarily among them.

To apply Lemma 2, we need a constant $Q'_{\min}$ such that $\min\{Q'_0, Q'_1\} \ge Q'_{\min}$. We present a straightforward choice based on the lower bound on the left-hand side of (10) (proved in Section III-C). By choosing $Q'_{\min}$ such that even the mutual information $I(X;Y)$ is upper bounded by the left-hand side of (10) when $\min\{Q'_0, Q'_1\} < Q'_{\min}$, we see from the simple identity $I_{\mathrm{LM}}(Q) \le I(X;Y)$ [3] that Q cannot maximize I_{LM}. For the example under consideration (see (9)), the choice $Q'_{\min} = 0.042$ turns out to be sufficient, and in fact yields $I(X;Y) \le 0.135$. This can be verified by computing $I(X;Y)$ to be (approximately) 0.0917, 0.4919 and 0.1348 for $Q_0 = 0.042$, $Q_0 = 0.5$ and $Q_0 = 1 - 0.042$ respectively, and then using the concavity of $I(X;Y)$ in Q to handle $Q_0 \in [0, 0.042) \cup (1 - 0.042, 1]$.

Let $h \triangleq 10^{-5}$, and suppose that we evaluate $I_{\mathrm{LM}}(Q)$ for each Q_0 in the set

$$\mathcal{A} \triangleq \big\{Q'_{\min}, Q'_{\min} + h, \dots, 1 - Q'_{\min} - h, 1 - Q'_{\min}\big\}. \quad (28)$$

Since the optimal input distribution Q' corresponds to some $Q'_0 \in [Q'_{\min}, 1 - Q'_{\min}]$, we conclude that there exists some $Q_0 \in \mathcal{A}$ such that $|Q'_0 - Q_0| \le \frac{h}{2}$. Substituting $\delta = \frac{h}{2} = 0.5 \times 10^{-5}$ and $Q'_{\min} = 0.042$ into (17), we conclude that

$$\max_{Q_0 \in \mathcal{A}} I_{\mathrm{LM}}(Q) \ge C_{\mathrm{LM}} - 0.982 \times 10^{-4}. \quad (29)$$

We now describe our techniques for evaluating $I_{\mathrm{LM}}(Q)$ for a fixed choice of Q. This is straightforward in principle, since the corresponding optimization problem is convex whether we use the primal expression in (3) or the dual expression in (4). Nevertheless, since we need to test a large number of Q_0 values, we make an effort to find a reasonably efficient method.

We avoid using the dual expression in (4), since it is a *maximization* problem; thus, if the final optimization parameters obtained differ slightly from the true optimal parameters, they will only provide a lower bound on $I_{\mathrm{LM}}(Q)$. In contrast, the result that we seek is an upper bound. We also avoid evaluating (3) directly, since the equality constraints in the optimization problem could, in principle, be sensitive to numerical precision errors.

Of course, there are many ways to circumvent these problems and provide rigorous bounds on the suboptimality of optimization procedures, including a number of generic solvers. We instead take a different approach, and reduce the primal optimization in (10) to a *scalar minimization* problem by eliminating the constraints one-by-one. This minimization will contain no equality constraints, and thus minor variations in the optimal parameter will still produce a valid upper bound.

We first note that the inequality constraint can be replaced by an equality whenever $I_{\mathrm{LM}}(Q) > 0$ [3, Lemma 1], which is certainly the case for the present example. Moreover, since the X-marginal is constrained to equal Q, we can let the minimization be over $\mathcal{P}(\mathcal{Y}|\mathcal{X})$ instead of $\mathcal{P}(\mathcal{X}\times\mathcal{Y})$, yielding

$$I_{\rm LM}(Q) = \min_{\substack{\widetilde{W}\in\mathcal{P}(\mathcal{Y}|\mathcal{X})\,:\,\widetilde{P}_Y=P_Y \\ \mathbb{E}_{Q\times\widetilde{W}}[\log q(X,Y)]=\mathbb{E}_P[\log q(X,Y)]}} I_{Q\times\widetilde{W}}(X;Y), \tag{30}$$

where $\widetilde{P}_Y(y) \triangleq \sum_x Q(x)\widetilde{W}(y|x)$ (recall also that $P_{XY} = Q\times W$). Let us fix a conditional distribution $\widetilde{W}$ satisfying the specified constraints, and write $\widetilde{W}_{xy} \triangleq \widetilde{W}(y|x)$. The analogous matrix to $\boldsymbol{W}$ in (9) can be written as follows:

$$\widetilde{\boldsymbol{W}} = \begin{bmatrix} \widetilde{W}_{00} & \widetilde{W}_{01} & 1-\widetilde{W}_{00}-\widetilde{W}_{01} \\ \widetilde{W}_{10} & \widetilde{W}_{11} & 1-\widetilde{W}_{10}-\widetilde{W}_{11} \end{bmatrix}. \tag{31}$$

Since $\widetilde{P}_Y = P_Y$ implies $H(\widetilde{P}_Y) = H(P_Y)$, we can write the objective in (30) as

$$\begin{aligned} &I_{Q\times\widetilde{W}}(X;Y) \\ &= H(P_Y) - H_{Q\times\widetilde{W}}(Y|X) \quad (32)\\ &= H(P_Y) + Q_0\big(\widetilde{W}_{00}\log\widetilde{W}_{00} + \widetilde{W}_{01}\log\widetilde{W}_{01} \\ &\qquad + (1-\widetilde{W}_{00}-\widetilde{W}_{01})\log(1-\widetilde{W}_{00}-\widetilde{W}_{01})\big) \\ &\qquad + Q_1\big(\widetilde{W}_{10}\log\widetilde{W}_{10} + \widetilde{W}_{11}\log\widetilde{W}_{11} \\ &\qquad + (1-\widetilde{W}_{10}-\widetilde{W}_{11})\log(1-\widetilde{W}_{10}-\widetilde{W}_{11})\big). \quad (33)\end{aligned}$$

We now show that the equality constraints can be used to express each $\widetilde{W}_{xy}$ in terms of $\widetilde{W}_{10}$. Using $\widetilde{P}_Y(y) = P_Y(y)$ for $y=0,1$, along with the constraint containing the decoding metric, we have

$$Q_0\widetilde{W}_{00} + Q_1\widetilde{W}_{10} = P_Y(0) \tag{34}$$

$$Q_0\widetilde{W}_{01} + Q_1\widetilde{W}_{11} = P_Y(1) \tag{35}$$

$$Q_1\big(\widetilde{W}_{11}\log q_{11} + (1-\widetilde{W}_{10}-\widetilde{W}_{11})\log q_{12}\big) = \mathbb{E}_P[\log q(X,Y)], \tag{36}$$

where in (36) we used the fact that $\log q(x,y) = 0$ for four of the six (x,y) pairs (see (9)). Re-arranging (34)–(36), we obtain

$$\widetilde{W}_{00} = \frac{P_Y(0) - Q_1\widetilde{W}_{10}}{Q_0} \tag{37}$$

$$\widetilde{W}_{01} = \frac{P_Y(1) - Q_1\widetilde{W}_{11}}{Q_0} \tag{38}$$

$$\widetilde{W}_{11} = \frac{1}{\log q_{11} - \log q_{12}} \times\left(\frac{\mathbb{E}_P[\log q(X,Y)]}{Q_1} - (1-\widetilde{W}_{10})\log q_{12}\right), \tag{39}$$

and substituting (39) into (38) yields

$$\widetilde{W}_{01} = \frac{1}{Q_0}\left(P_Y(1) - \frac{1}{\log q_{11} - \log q_{12}} \times\left(\mathbb{E}_P[\log q(X,Y)] - Q_1(1-\widetilde{W}_{10})\log q_{12}\right)\right). \tag{40}$$

We have thus written each entry of (33) in terms of $\widetilde{W}_{10}$, and we are left with a one-dimensional optimization problem. However, we must still ensure that the constraints $\widetilde{W}_{xy}\in[0,1]$ are satisfied for all (x,y). Since each $\widetilde{W}_{xy}$ is an affine function of $\widetilde{W}_{10}$, these constraints are each of the form $\underline{W}^{(x,y)} \le \widetilde{W}_{10} \le \overline{W}^{(x,y)}$, and the overall optimization is given by

$$\min_{\underline{W}\le\widetilde{W}_{10}\le\overline{W}} f(\widetilde{W}_{10}), \tag{41}$$

where $f(\cdot)$ denotes the right-hand side of (33) upon substituting (37), (39) and (40), and the lower and upper limits are given by $\underline{W} \triangleq \max_{x,y}\underline{W}^{(x,y)}$ and $\overline{W} \triangleq \min_{x,y}\overline{W}^{(x,y)}$. Note that the minimization region is non-empty, since $\widetilde{\boldsymbol{W}} = \boldsymbol{W}$ is always feasible. In principle one could observe $\underline{W} = \overline{W} = W_{10}$, but in the present example we found that $\underline{W} < \overline{W}$ for every choice of Q_0 that we used.

The optimization problem in (41) does not appear to permit an explicit solution. However, we can efficiently compute the solution to high accuracy using standard one-dimensional optimization methods. Since the convexity of any optimization problem is preserved by the elimination of equality constraints [17, Sec. 4.2.4], and since the optimization problem in (30) is convex for any given Q, we conclude that $f(\cdot)$ is a convex function. Its derivative is easily computed by noting that

$$\frac{d}{dz}(\alpha z+\beta)\log(\alpha z+\beta) = \alpha + \alpha\log(\alpha z+\beta) \tag{42}$$

for all α, β and z yielding a positive argument to the logarithm. We can thus perform a bisection search as follows, where $f'(\cdot)$ denotes the derivative of f, and ϵ is a termination parameter:

1) Set $i=0$, $\underline{W}^{(0)} = \underline{W}$ and $\overline{W}^{(0)} = \overline{W}$;
2) Set $W_{\rm mid} = \frac{1}{2}(\underline{W}^{(i)} + \overline{W}^{(i)})$; if $f'(W_{\rm mid}) \ge 0$ then set $\underline{W}^{(i+1)} = \underline{W}^{(i)}$ and $\overline{W}^{(i+1)} = W_{\rm mid}$; otherwise set $\underline{W}^{(i+1)} = W_{\rm mid}$ and $\overline{W}^{(i+1)} = \overline{W}^{(i)}$;
3) If $|f'(W_{\rm mid})| \le \epsilon$ then terminate; otherwise increment i and return to Step 2.

As mentioned previously, we do not need to find the exact solution to (41), since any value of $\widetilde{W}_{10}\in[\underline{W},\overline{W}]$ yields a valid upper bound on $I_{\rm LM}(Q)$. However, we must choose ϵ sufficiently small so that the bound in (10) is established. We found $\epsilon = 10^{-6}$ to suffice.

We implemented the preceding techniques in C (see [15] for the code) to upper bound $I_{\rm LM}(Q)$ for each $Q_0\in\mathcal{A}$; see Figure 1. As stated following Counter-Example 1, we found the highest value of $I_{\rm LM}(Q)$ to be the right-hand side of (12), corresponding to the input distribution in (13). We found the corresponding minimizing parameter in (41) to be roughly $\widetilde{W}_{10} = 0.4252347$.

Instead of directly adding 10^{-4} to (12) in accordance with (29), we obtain a refined estimate by "updating" our estimate of $Q'_{\min}$. Specifically, using (29) and observing the values in Figure 1, we can conclude that the optimal value of Q_0 lies in the range $[0.7, 0.8]$ (we are being highly conservative here). Thus, setting $Q'_{\min} = 0.2$ and using the previously chosen value $\delta = 0.5\times10^{-5}$, we obtain the following refinement of (29):

$$\max_{Q_0\in\mathcal{A}} I_{\rm LM}(Q) \ge C_{\rm LM} - 2.43\times10^{-5}. \tag{43}$$

Since our implementation in C is based on floating-point calculations, the final values may have precision errors. We therefore checked our numbers using Mathematica's arbitrary-precision arithmetic framework [18], which allows one to work with *exact* expressions that can then be displayed to arbitrarily many decimal places. More precisely, we loaded the values of $\widetilde{W}_{10}$ into Mathematica and rounded them to 12 decimal places (this is allowed, since any value of $\widetilde{W}_{10}$ yields a valid upper bound). Using the exact values of all other quantities (e.g. Q and W), we performed an evaluation of $f(\widetilde{W}_{10})$ in (41), and compared it to the corresponding value of $I_{\rm LM}(Q)$ produced by the C program. The maximum discrepancy across all of the values of Q_0 was less than 2.1×10^{-12}. Our final bound in (10) was obtained by adding 2.5×10^{-5} (which is, of course, higher than $2.43 \times 10^{-5} + 2.1 \times 10^{-12}$) to the right-hand side of (12).

C. Establishing the Lower Bound in (10)

For the lower bound, we can afford to be less careful than we were in establishing the upper bound; all we need is a suitable choice of Q and the parameters (s, a) in (4). We choose Q as in (13), along with the following:

$$s = 9.031844 \tag{44}$$

$$\boldsymbol{a} = \left[\begin{array}{cc} 0.355033 & -0.355033 \end{array}\right], \tag{45}$$

In [11, Appendix A], we provide details on how these parameters were obtained, though the desired lower bound can readily be verified without knowing such details.

Using these values, we evaluated the objective in (4) using Mathematica's arbitrary-precision arithmetic framework [18], thus eliminating the possibility of arithmetic precision errors. See [15] for the relevant C and Mathematica code.

D. Establishing the Lower Bound in (11)

We establish the lower bound in (11) by setting $\mathcal{U} = \{0, 1\}$ and forming a suitable choice of Q_{UX}, and then using the dual expressions in (7)–(8) to lower bound $I_{\rm SC}^{(2)}(Q_{UX})$.

1) Choice of Input Distribution: Let $Q = [Q_0 \ Q_1]$ be some input distribution on $\mathcal{X}$, and define the corresponding product distribution on $\mathcal{X}^2$ as

$$Q^{(2)} = \left[\begin{array}{cccc} Q_0^2 & Q_0 Q_1 & Q_0 Q_1 & Q_1^2 \end{array}\right], \tag{46}$$

where the order of the inputs is $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. Consider now the following choice of superposition coding parameters for the second-order product channel $(W^{(2)}, q^{(2)})$:

$$Q_U = \left[\begin{array}{cc} 1 - Q_1^2 & Q_1^2 \end{array}\right] \tag{47}$$

$$Q_{X|U=0} = \frac{1}{1 - Q_1^2}\left[\begin{array}{cccc} Q_0^2 & Q_0 Q_1 & Q_0 Q_1 & 0 \end{array}\right] \tag{48}$$

$$Q_{X|U=1} = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right]. \tag{49}$$

This choice yields an X-marginal Q_X precisely given by (46), and it is motivated by the empirical observation from [6] that choices of Q_{UX} where $Q_{X|U=1}$ and $Q_{X|U=2}$ have disjoint supports tend to provide good rates. We let the single-letter distribution $Q = [Q_0 \ Q_1]$ be

$$Q = \left[\begin{array}{cc} 0.749 & 0.251 \end{array}\right]. \tag{50}$$

which we chose based on a simple brute force search (see Figure 1). Note that this choice is similar to that in (13), but not identical.

One may question whether the choice of the supports of $Q_{X|U=0}$ and $Q_{X|U=1}$ in (48)–(49) is optimal. For example, a similar construction might set $Q_U(0) = Q_0^2 + Q_0 Q_1$, and then replace (48)–(49) by normalized versions of $[Q_0^2 \ Q_0 Q_1 \ 0 \ 0]$ and $[0 \ 0 \ Q_0 Q_1 \ Q_1^2]$. However, after performing a brute force search over the possible support patterns (there are no more than 2^4, and many can be ruled out by symmetry considerations), we found the above pattern to be the only one to give an improvement on $I_{\rm LM}$, at least for the choices of input distribution in (13) and (50). In fact, even after setting $|\mathcal{U}| = 3$, considering the third-order product channel $(W^{(3)}, q^{(3)})$, and performing a similar brute force search over the support patterns (of which there are no more than 3^8), we were unable to obtain an improvement on (11).

2) Choices of Optimization Parameters: We now specify the choices of the dual parameters in (7)–(8). In [11, Appendix A], we give details of how these parameters were obtained. We claim that the choice

$$(R_0, R_1) = (0.0356005, 0.2403966) \tag{51}$$

is permitted; observe that summing these two values and dividing by two (since we are considering the product channel) yields (11). These values can be verified by setting the parameters as follows: On the right-hand side of (7), set

$$s = 9.4261226 \tag{52}$$

$$\boldsymbol{a} = \left[\begin{array}{cccc} 0.4817048 & -0.2408524 & -0.2408524 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \tag{53}$$

and on the right-hand side of (8), set

$$\rho_1 = 0.7587516 \tag{54}$$

$$s = 9.3419338 \tag{55}$$

$$\boldsymbol{a} = \left[\begin{array}{cccc} 0.7186926 & -0.0488036 & -0.0488036 & 0 \\ 0 & 0 & 0 & -0.6210855 \end{array}\right]. \tag{56}$$

Once again, we evaluated (7)–(8) using Mathematica's arbitrary-precision arithmetic framework [18], thus ensuring the validity of (11). See [15] for the relevant C and Mathematica code.

IV. Conclusion

We have used our numerical findings, along with an analysis of the gap to suboptimality for slightly suboptimal input distributions, to show that it is possible for $C_{\rm M}$ to exceed $C_{\rm LM}$ even for binary-input mismatched DMCs. This is in contrast with the claim in [14] that $C_{\rm M} = C_{\rm LM}$ for such channels.

An interesting direction for future research is to find a purely theoretical proof of Counter-Example 1; the non-concavity of $I_{\rm LM}(Q)$ observed in Figure 1 may play a role in such an investigation. Furthermore, it would be of significant interest to develop a better understanding of [14], including which parts may be incorrect, under what conditions the converse remains valid, and in the remaining cases, whether a valid converse lying in between the LM rate and matched capacity can be inferred.

Appendix

Achieving (11) via Expurgated Parallel Coding

Here we outline how the achievable rate of 0.137998 nats/use in (11) can be obtained using Lapidoth's expurgated parallel coding rate. We verified this value by evaluating the primal expressions in [9] using CVX [19], and also by evaluating the equivalent dual expressions in [6] by a suitable adaptation of the dual optimization parameters for superposition coding given in Section III-D. Here we focus on the latter, since it immediately provides a concrete lower bound even when the optimization parameters are suboptimal.

The parameters to Lapidoth's rate are two finite alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, two corresponding input distributions Q_1 and Q_2, and a function $\phi(x_1, x_2)$ mapping $\mathcal{X}_1$ and $\mathcal{X}_2$ to the channel input alphabet. For any such parameters, the rate $R = R_1 + R_2$ is achievable provided that [6], [8]

$$R_1 \le \sup_{s\ge 0, a(\cdot,\cdot)} \mathbb{E}\left[\log \frac{q(\phi(X_1,X_2),Y)^s e^{a(X_1,X_2)}}{\mathbb{E}\big[q(\phi(\overline{X}_1,X_2),Y)^s e^{a(\overline{X}_1,X_2)} \,\big|\, X_2, Y\big]}\right] \tag{57}$$

$$R_2 \le \sup_{s\ge 0, a(\cdot,\cdot)} \mathbb{E}\left[\log \frac{q(\phi(X_1,X_2),Y)^s e^{a(X_1,X_2)}}{\mathbb{E}\big[q(\phi(X_1,\overline{X}_2),Y)^s e^{a(X_1,\overline{X}_2)} \,\big|\, X_1, Y\big]}\right], \tag{58}$$

and at least one of the following holds:

$$R_1 \le \sup_{\rho_2\in[0,1], s\ge 0, a(\cdot,\cdot)} -\rho_2 R_2 + \mathbb{E}\left[\log \frac{\big(q(\phi(X_1,X_2),Y)^s e^{a(X_1,X_2)}\big)^{\rho_2}}{\mathbb{E}\Big[\Big(\mathbb{E}\big[q(\phi(\overline{X}_1,\overline{X}_2),Y)^s e^{a(\overline{X}_1,\overline{X}_2)} \,\big|\, \overline{X}_1\big]\Big)^{\rho_2} \,\Big|\, Y\Big]}\right] \tag{59}$$

$$R_2 \le \sup_{\rho_1\in[0,1], s\ge 0, a(\cdot,\cdot)} -\rho_1 R_1 + \mathbb{E}\left[\log \frac{\big(q(\phi(X_1,X_2),Y)^s e^{a(X_1,X_2)}\big)^{\rho_1}}{\mathbb{E}\Big[\Big(\mathbb{E}\big[q(\phi(\overline{X}_1,\overline{X}_2),Y)^s e^{a(\overline{X}_1,\overline{X}_2)} \,\big|\, \overline{X}_2\big]\Big)^{\rho_1} \,\Big|\, Y\Big]}\right], \tag{60}$$

where $(X_1, X_2, Y, \overline{X}_1, \overline{X}_2)$ are distributed according to $Q_1(x_1)Q_2(x_2)W(y|\phi(x_1,x_2))Q_1(\overline{x}_1)Q_2(\overline{x}_2)$.

Recall the input distribution Q_{UX} for superposition coding on the second-order product channel given in (47)–(49). Denoting the four inputs of the product channel as $\{(0,0),(0,1),(1,0),(1,1)\}$, we set $\mathcal{X}_1 = \{(0,0),(0,1),(1,0)\}$, $\mathcal{X}_2 = \mathcal{U} = \{0,1\}$, and

$$Q_{X_1} = \frac{1}{1-Q_1^2}\left[\begin{array}{ccc} Q_0^2 & Q_0Q_1 & Q_0Q_1 \end{array}\right] \tag{61}$$

$$Q_{X_2} = \left[\begin{array}{cc} 1-Q_1^2 & Q_1^2 \end{array}\right] \tag{62}$$

$$\phi(x_1,x_2) = \begin{cases} x_1 & x_2 = 0 \\ (1,1) & x_2 = 1. \end{cases} \tag{63}$$

This induces a joint distribution $Q_{X_1X_2X}(x_1,x_2,x) = Q_{X_1}(x_1)Q_{X_2}(x_2)\mathbb{1}\{x = \phi(x_1,x_2)\}$. The idea behind this choice is that the marginal distribution Q_{X_2X} coincides with our choice of Q_{UX} for SC.

By the structure of our input distributions, there is in fact a one-to-one correspondence between (u,x) and (x_1,x_2), thus allowing us to immediately use the dual parameters (s,a,ρ_1) from SC for the expurgated parallel coding rate. More precisely, using the superscripts $(\cdot)^{\mathrm{sc}}$ and $(\cdot)^{\mathrm{ex}}$ to distinguish between the two ensembles, we set

$$R_1^{\mathrm{ex}} = R_1^{\mathrm{sc}} \tag{64}$$

$$R_2^{\mathrm{ex}} = R_0^{\mathrm{sc}} \tag{65}$$

$$s^{\mathrm{ex}} = s^{\mathrm{sc}} \tag{66}$$

$$a^{\mathrm{ex}}(x_1,x_2) = a^{\mathrm{sc}}(x_2,\phi(x_1,x_2)) \tag{67}$$

$$\rho_1^{\mathrm{ex}} = \rho_1^{\mathrm{sc}}. \tag{68}$$

Using these identifications along with the choices of the superposition coding parameters in (52)–(56), we verified numerically that the right-hand side of (57) (respectively, (60)) coincides with that of (7) (respectively, (8)). Finally, to conclude that the expurgated parallel coding rate recovers (11), we numerically verified that the rate R_2 resulting from (57) and (60) (which, from (51), is 0.0356005) also satisfies (58). In fact, the inequality is strict, with the right-hand side of (58) being at least 0.088.

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